

ISOMETRIES OF OPTIMAL PSEUDO-RIEMANNIAN METRICS

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ABSTRACT. We give a concise proof that large classes of optimal (constant curvature or Einstein) pseudo-Riemannian metrics are maximally symmetric within their conformal class.

The study of optimal metrics has held a central position in the study of Riemannian and pseudo-Riemannian geometry. In this note, we give a very short alternative proof—for some classes of special cases—of general results of Hebey–Vaugon [HV93] and Ferrand [Fer96] on the symmetry of optimal Riemannian metrics. Via the same short proof, we also extend these results to some pseudo-Riemannian and non-compact cases. In particular, we show that many optimal (constant curvature or Einstein) pseudo-Riemannian metrics are, in a very strong sense, the most symmetric metrics within their conformal class. The general result says that if this optimal metric is (after an appropriate normalization) unique within its conformal class, then any conformal transformation (and, in particular, isometry) of a given pseudo-Riemannian metric in that class must be an isometry of the optimal metric. We then give some circumstances in which this general result applies.

Throughout the paper, we assume all manifolds are connected and, for simplicity, that they are smooth. They may be open, closed, or with boundary. Also for simplicity, we assume all Riemannian metrics are smooth, though they may be complete or incomplete unless otherwise mentioned.

1. GENERAL RESULTS

We begin with the following definition.

Definition 1. Let \mathcal{G} be a collection of pseudo-Riemannian metrics on a manifold M , and let $g_0 \in \mathcal{G}$. We say g_0 is *strongly maximally symmetric within \mathcal{G}* if, for all $g \in \mathcal{G}$ and for each isometry $\varphi \in \text{Diff}(M)$ of g , φ is an isometry of g_0 .

Let's denote the isometry group of a metric g by I_g . An obviously equivalent formulation of the definition is the following. A metric g_0 which is strongly maximally symmetric within \mathcal{G} has the property that, for any $g \in \mathcal{G}$, I_g is a subgroup of I_{g_0} when both are considered as subgroups of $\text{Diff}(M)$.

Our main theorem can now be stated in full generality as follows. Consider the set \mathcal{M} of all Riemannian metrics on M , and let $\mathcal{N} \subseteq \mathcal{M}$ be any $\text{Diff}(M)$ -invariant subset. (For example, \mathcal{N} could be all metrics with constant scalar curvature -1 .) For any metric g on M , we denote by C_g the group of conformal diffeomorphisms of (M, g) , that is, the set of $\varphi \in \text{Diff}(M)$ with $\varphi^*g = \sigma g$ for some $\sigma \in C^\infty(M)$.

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Theorem 2. *Let a conformal class \mathcal{C} of pseudo-Riemannian metrics on a manifold M be given. If there is a unique metric $g_0 \in \mathcal{C} \cap \mathcal{N}$, then for each $g \in \mathcal{C}$, C_g is a subgroup of I_{g_0} . In particular, g_0 is strongly maximally symmetric within \mathcal{C} .*

Proof. For any function $\rho \in C^\infty(M)$, let $g := e^\rho g_0 \in \mathcal{C}$. Let $\varphi \in C_g$ be any conformal diffeomorphism of g , say $\varphi^*g = e^\sigma g$. Then we have

$$(1) \quad \varphi^*g_0 = \varphi^*(e^{-\rho}g) = \varphi^*(e^{-\rho})(\varphi^*g) = (\varphi^*e^\rho)^{-1}e^\sigma g.$$

Therefore φ^*g_0 is conformal to g_0 . On the other hand, since \mathcal{N} is diffeomorphism-invariant, $\varphi^*g_0 \in \mathcal{N}$. But by assumption, g_0 is the only metric in $\mathcal{C} \cap \mathcal{N}$, so in fact $\varphi^*g_0 = g_0$, i.e., $\varphi \in I_{g_0}$. \square

Remark 3. The reader should compare Theorem 2 (and its corollaries below) with Ferrand's proof of the Lichnerowicz conjecture [Fer96, Thm. A] (which was proved by Obata [Oba72] in the compact case for the connected component of the identity in C_g , and which can be proved via a simpler argument for $\dim M = 2$) stating that if $\dim M \geq 3$ and a Riemannian manifold (M, g) is not conformally equivalent to a round sphere or Euclidean space, then C_g is *inessential*. That is, it is contained in the isometry group of some metric g_0 in the conformal class of g . We note that these results do not explicitly identify the metric g_0 .

In particular, one should also compare Theorem 2 to the resolution of the equivariant Yamabe problem by Hebey–Vaugon [HV93], which uses a different strategy to show that for any compact Riemannian manifold (M, g) ($\dim M \geq 3$), there is an I_g -invariant metric g_0 that is conformal to g and has constant scalar curvature.

The primary differences between our result and above-mentioned results are threefold. First, our proof is more concise. Second, it is applicable to certain pseudo-Riemannian cases not handled in [Fer96, HV93] (however, one should compare [FM10]), and certain noncompact cases not handled in [HV93]. Third, our proof is deficient in that it only solves the Lichnerowicz conjecture or the equivariant Yamabe problem in dimension two (cf. §2.1; the latter is also solved in dimension two via our method only when assuming $M \neq S^2$), while only partially recovering these results in dimension three and higher (cf. §2.2 and §2.3).

For any function ζ on M , let $I_\zeta = \{\varphi \in \text{Diff}(M) \mid \varphi^*\zeta = \zeta\}$ denote its isotropy group.

Working along the same lines as the proof of Theorem 2, we get the following characterization of the isometries of a metric in the conformal class of a strongly maximally symmetric metric.

Theorem 4. *Let a conformal class \mathcal{C} of pseudo-Riemannian metrics on a manifold M be given. If $g_0 \in \mathcal{C}$ is strongly maximally symmetric within \mathcal{C} and $g = \rho g_0$ for some positive function ρ on M , then $I_g = I_\rho \cap I_{g_0}$.*

Proof. If $\varphi \in I_g$, then by assumption $\varphi \in I_{g_0}$. Similarly to (1), we have

$$\rho^{-1}g = g_0 = \varphi^*g_0 = (\varphi^*\rho)^{-1}g,$$

which implies that $\varphi^*\rho = \rho$. \square

2. APPLICATIONS

Let us now give some examples of subsets $\mathcal{N} \subseteq \mathcal{M}$ for which there are conformal classes containing a unique metric in \mathcal{N} , so that the theorems of the previous section apply. We will give the statements corresponding to Theorem 2; the statements for Theorem 4 are of course analogous. In this section, we assume $\partial M = \emptyset$.

2.1. Constant Gaussian curvature. Let M be two-dimensional, and let g be any Riemannian metric on M . By Poincaré uniformization, there exists a metric g_0 that is conformal to g and is complete with constant Gaussian curvature η , where $\eta \in \{+1, 0, -1\}$. If $M \cong S^2$ (i.e., $\eta = +1$), then g_0 is not unique. If $\eta = 0$, then g_0 is unique only up to homothety if it is isometric to the Euclidean plane. On the other hand, (M, g_0) is unique if it is isometric to a cylinder or a torus if we additionally require $\text{inj}(M, g_0) = 1$. If $\eta = -1$, then g_0 is unique. If $\eta = -1$ and M is compact (i.e., a surface of genus $p \geq 2$), then Hurwitz's Theorem [FK92, p. 258] says that the order of I_{g_0} is finite and bounded above by $84(p-1)$.

Thus, we may in this case let \mathcal{N} be the subset of metrics g_0 with constant Gaussian curvature. If the curvature of g_0 is 0 and g_0 is not the Euclidean metric on the plane, then we additionally require $\text{inj}(M, g_0) = 1$. We obtain the following corollary of Theorem 2.

Corollary 5. *Let (M, g) be any smooth Riemannian 2-manifold, and let g_0 be any metric with constant Gaussian curvature conformal to g . If (M, g_0) is not isometric to the sphere with its round metric or the Euclidean plane, then any conformal diffeomorphism of g is an isometry of g_0 .*

Furthermore, if M is compact with genus $p \geq 2$, then the orders of C_g and I_g are finite and bounded above by $84(p-1)$. In particular, (M, g) admits no nontrivial conformal Killing fields.

Remark 6. The previous corollary also follows from a result of Calabi [Cal85, Thm. 3] (cf. also [Lic57, Mat57]). One can also deduce the previous corollary using the fact that isometries of the constant-curvature metric on a surface (other than S^2 and \mathbb{C}) agree with biholomorphisms of the complex structure. Again, the advantage of our proof is its simplicity.

2.2. Constant scalar curvature. For the rest of the paper, we consider manifolds M with $\dim M \geq 3$. If (M, g) is a compact Riemannian manifold, then by Trudinger, Aubin, and Schoen's resolution of the Yamabe problem [Tru68, Aub76, Sch84], there exists a metric g_0 conformal to g of constant scalar curvature.

In general, the metric g_0 need not be the unique (even up to homothety) metric conformal to g of constant scalar curvature, so that Theorem 2 does not necessarily apply. However, if the scalar curvature of g_0 is nonpositive, then a maximum principle argument implies that g_0 is, in fact, unique up to homothety [KW75, Thm. 4.2]. Therefore, letting \mathcal{N} be the subset of metrics on M with unit volume and constant scalar curvature, we obtain:

Corollary 7. *Let (M, g) be a smooth, compact Riemannian manifold with nonpositive Yamabe invariant. Then any conformal diffeomorphism of g is an isometry of the unique metric g_0 conformal to g with constant scalar curvature and unit volume.*

2.3. Einstein metrics. We now allow the case where (M, g) is pseudo-Riemannian, and not just Riemannian. Using a result of Kühnel and Rademacher [KR95, Thm. 1*], we can show the following. We assume that the number of negative eigenvalues in the signature of g is no greater than $\frac{n}{2}$. The results of this subsection apply even if g is only of regularity C^3 .

If there is an Einstein metric g_0 in the conformal class of g , then this metric is unique up to homothety, except in the following two cases:

- (1) (M, g_0) is a simply-connected Riemannian space of constant sectional curvature.
- (2) (M, g_0) is a warped-product manifold $\mathbb{R} \times_{e^{2t}} N$, where N is a complete Ricci-flat $(n-1)$ -dimensional Riemannian manifold ($n = \dim M$). Explicitly, the metric is

given by $\pm dt^2 + e^{2t}h$, where h is the metric on N and the sign of dt^2 depends on the signature of g_0 .

If neither of the above cases holds, we can choose the Einstein metric g_0 uniquely (i.e., fix a scale) in some circumstances, e.g., the following:

- (a) If the scalar curvature of g_0 is positive or negative, scale so that it is ± 1 .
- (b) If $\text{Vol}(M, g_0) < \infty$, scale to unit volume.
- (c) If $0 < \text{inj}(M, g_0) < \infty$, scale so that $\text{inj}(M, g_0) = 1$.

Thus, letting \mathcal{N} be the set of such metrics, we have the following:

Corollary 8. *Let (M, g) be a pseudo-Riemannian manifold of regularity at least C^3 , with $\dim M \geq 3$. Assume that g has no greater than $\frac{n}{2}$ negative eigenvalues. If (M, g) is conformally equivalent to a (geodesically) complete Einstein manifold (M, g_0) , for which neither case (1) nor (2) above holds, and for which either (a), (b), or (c) applies, then every conformal diffeomorphism of g is an isometry of g_0 .*

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